



2.4 Other vector spaces





Introduction

- In this topic, we will describe the vector spaces of
 - Discrete signals
 - Polynomials
 - Continuous signals
 - Images





Discrete signals

- A discrete signal is an infinite sequence of real or complex values
 - These are generally represented by italicized lower-case x, y and z
$$x = (37.2, 37.6, 37.5, 37.2, 36.9, 37.2, 37.5, 37.9, 38.1, 37.8, \dots)$$
- In general, the discrete signal x is represented by
$$x = (x[0], x[1], x[2], x[3], x[4], \dots)$$
- Such signals are readings from periodically sampled sensors or other periodically sampled data
 - Once you turn on a system,
you may not know when it will be turned off





Discrete signals

- Thus, we may define:
 - Scalar multiplication: given x , αx is the discrete signal
$$\alpha x = (\alpha x[0], \alpha x[1], \alpha x[2], \alpha x[3], \dots)$$
 - Vector addition: given x and y , $x + y$ is the discrete signal
$$x + y = (x[0] + y[0], x[1] + y[1], x[2] + y[2], x[3] + y[3], \dots)$$
 - The zero vector is
$$\hat{0} = (0, 0, 0, 0, 0, \dots)$$
 - Additive inverses: given x , $-x$ is the discrete signal
$$-x = (-x[0], -x[1], -x[2], -x[3], \dots)$$





Discrete signals

- Claim:
 - With these definitions of scalar multiplication, vector addition, the zero vector and additive inverses, the set of all discrete signals forms a real or complex vector space
- This shouldn't surprise you:
 - If a one-million-dimensional vector is a vector, there's no reason an infinite sequence cannot be a vector, too





Modelling discrete signals

- If $|\gamma| \leq 1$, then the signal

$$\gamma^n = (1, \gamma, \gamma^2, \gamma^3, \gamma^4, \dots)$$

$$\gamma^n[k] = \gamma^k$$

- This is a geometric sequence going to zero
- For example, $\frac{1}{2}^n = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots)$
- Thus $0.4\frac{1}{2}^n = (0.4, 0.2, 0.1, 0.05, 0.025, \dots)$
- Also, $\frac{1}{2}^n + -\frac{1}{2}^n = (2, 0, \frac{1}{2}, 0, \frac{1}{8}, \dots)$





Polynomials

- A polynomial in x is a linear combination of terms $1, x, x^2, x^3, x^4, \dots$
 - These are generally represented by italicized lower-case p, q and r
 - For example, p may be a polynomial and

$$p(x) = x^5 + 0.3x^2 + 0.5x + 0.9$$

- The degree of the polynomial p , denoted $\deg(p)$, is the largest exponent of a term within the polynomial with a non-zero coefficient

$$\deg(p) = 5$$

- Polynomials are used to locally approximate real-world continuous functions





Polynomials

- Given the polynomial

$$p(x) = x^5 + 0.2x^2 + 0.5x + 0.9$$

the value of this polynomial at x is simply found by evaluating the polynomial at that point; e.g., $p(1.5)$

- In general, an arbitrary polynomial p is represented by

$$p(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \cdots + \alpha_2 x^2 + \alpha_1 x + \alpha_0$$

$$= \sum_{k=0}^n \alpha_k x^k$$

- If we ask what α_n is, and $n > \deg(p)$, then $\alpha_n = 0$





Polynomials

- Thus, we may define:

- Scalar multiplication: given p , βp is the polynomial

$$(\beta p)(x) = \beta \alpha_n x^n + \beta \alpha_{n-1} x^{n-1} + \cdots + \beta \alpha_2 x^2 + \beta \alpha_1 x + \beta \alpha_0$$

- Vector addition: given p and q , $p + q$ is the polynomial

$$(p + q)(x) = (\alpha_m + \beta_m) x^m + (\alpha_{m-1} + \beta_{m-1}) x^{m-1} + \cdots \\ + (\alpha_2 + \beta_2) x^2 + (\alpha_1 + \beta_1) x + (\alpha_0 + \beta_0)$$

where $m = \deg(p + q) \leq \max \{ \deg(p), \deg(q) \}$

- The *zero vector* is the zero polynomial $\hat{0}$ where

$$\hat{0}(x) = 0$$

- Additive inverses: given p , $-p$ is the polynomial

$$(-p)(x) = -\alpha_n x^n - \alpha_{n-1} x^{n-1} - \cdots - \alpha_2 x^2 - \alpha_1 x - \alpha_0$$





Polynomials

- All polynomials are a linear combination of monomials x^n with coefficients equal to 1:

$$x^n(x) = x^n$$

- For example, if $p(x) = x^5 + 0.2x^2 + 0.5x + 0.9$
then $p = x^5 + 0.2x^2 + 0.5x + 0.9\hat{1}$





Polynomials

- Claim:
 - With these definitions of scalar multiplication, vector addition, the zero vector and additive inverses, the set of all polynomials in the variable x forms a real or complex vector space
- Engineers often represent the vector space of all polynomials in the variable x as $\mathbf{R}[x]$ or $\mathbf{C}[x]$, depending on whether the coefficients are real or complex, respectively





Continuous signals

- Next, we will look at continuous functions
 - We're however, interested in those functions that describe the real world
 - Thus, we will refer to such functions as *continuous signals*

- Given the continuous signal f where

$$f(t) = 0.53e^{-t} - 0.97e^{-2t}$$

the entries are accessed by $f(t)$ for some value of t in the domain of the non-negative reals





Continuous signals

- A continuous signal is a continuous function defined on some fixed domain D where that domain could be:
 - All real values
 - All reals greater than or equal to zero
 - All real values on an interval $[a, b]$

- These are generally represented
by italicized lower-case f , g and h

$$f(t) = 3.723e^{-0.2t}$$

- Such signals are actual real-world measurable values





Continuous signals

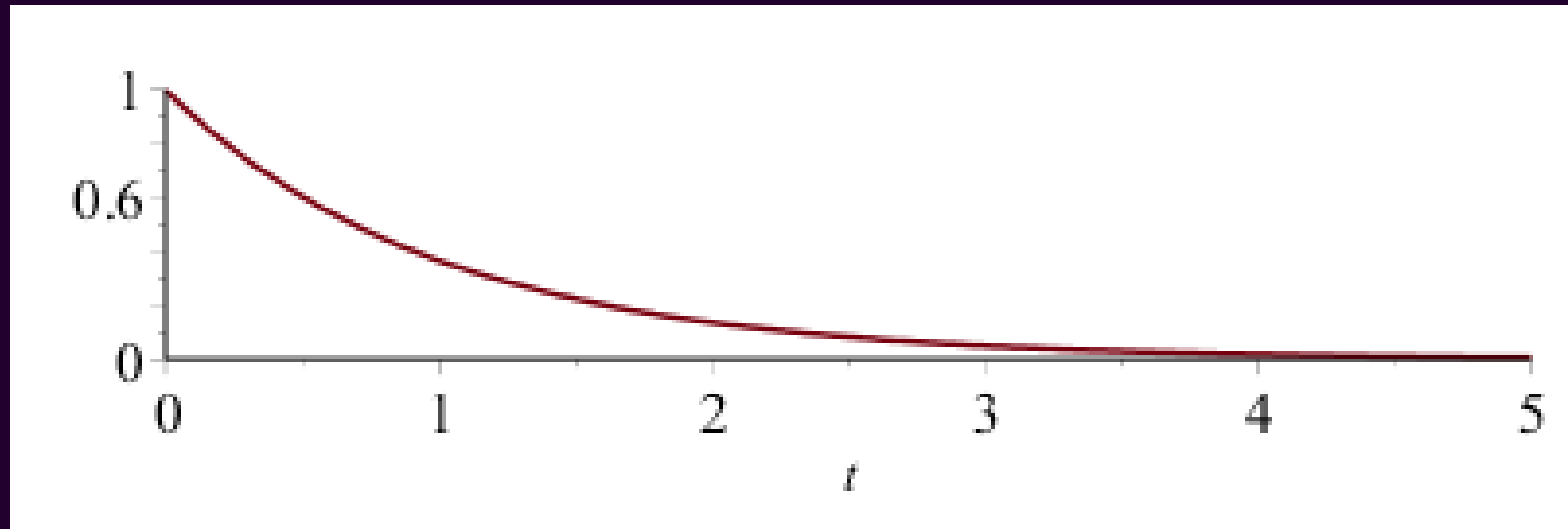
- Thus, we may define:
 - Scalar multiplication: given f , αf is the continuous signal
$$(\alpha f)(t) = \alpha f(t)$$
 - Vector addition: given f and g , $f + g$ is the continuous signal
$$(f + g)(t) = f(t) + g(t)$$
 - The zero vector is the zero function $\hat{0}$
$$\hat{0}(t) = 0$$
 - Additive inverses: given f , $-f$ is the continuous signal
$$(-f)(t) = -f(t)$$





Continuous signals

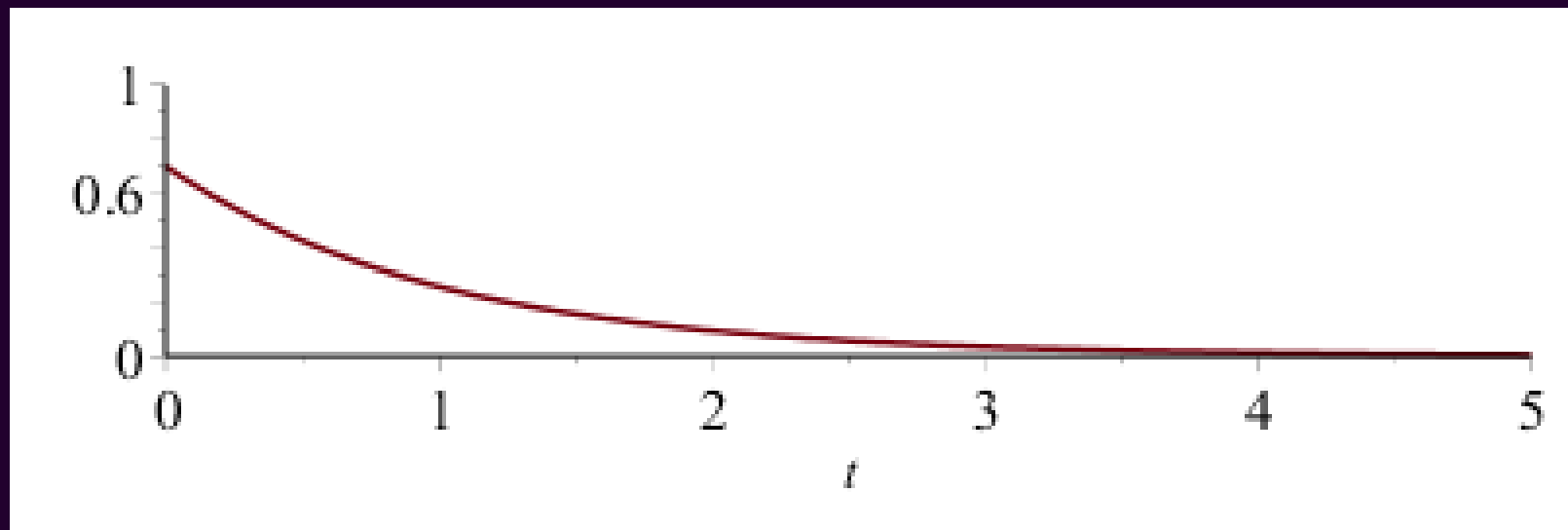
- For example, here is a signal f where $f(t) = e^{-t}$





Continuous signals

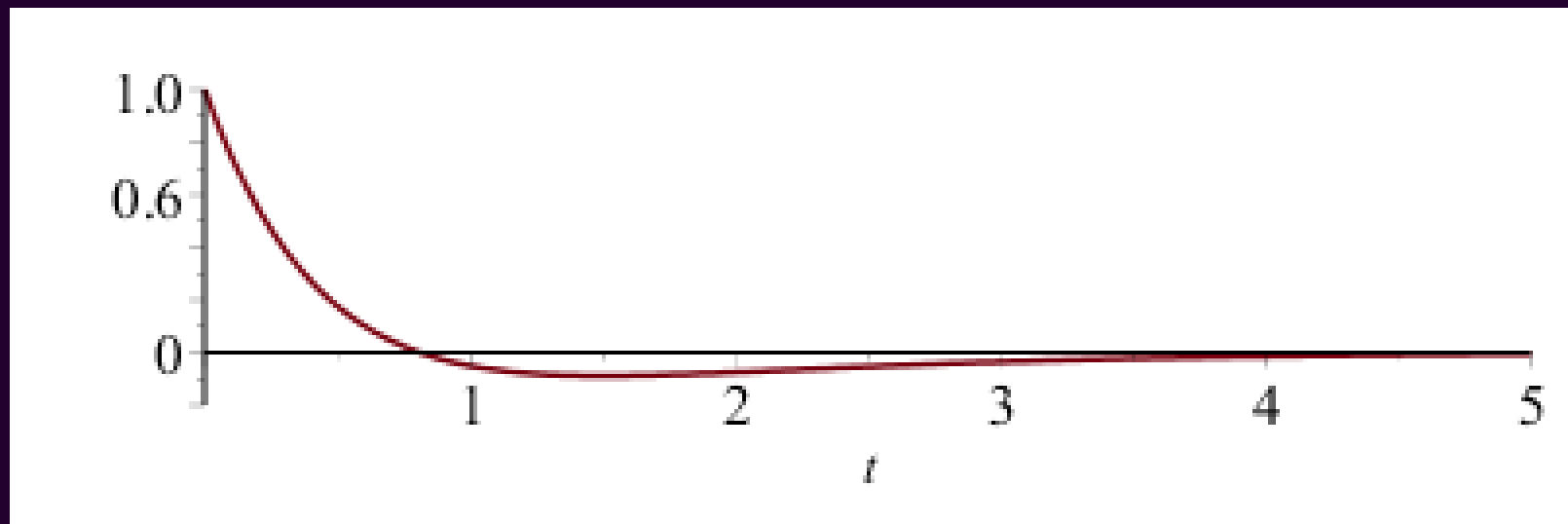
- Here is that signal multiplied by the scalar 0.7





Continuous signals

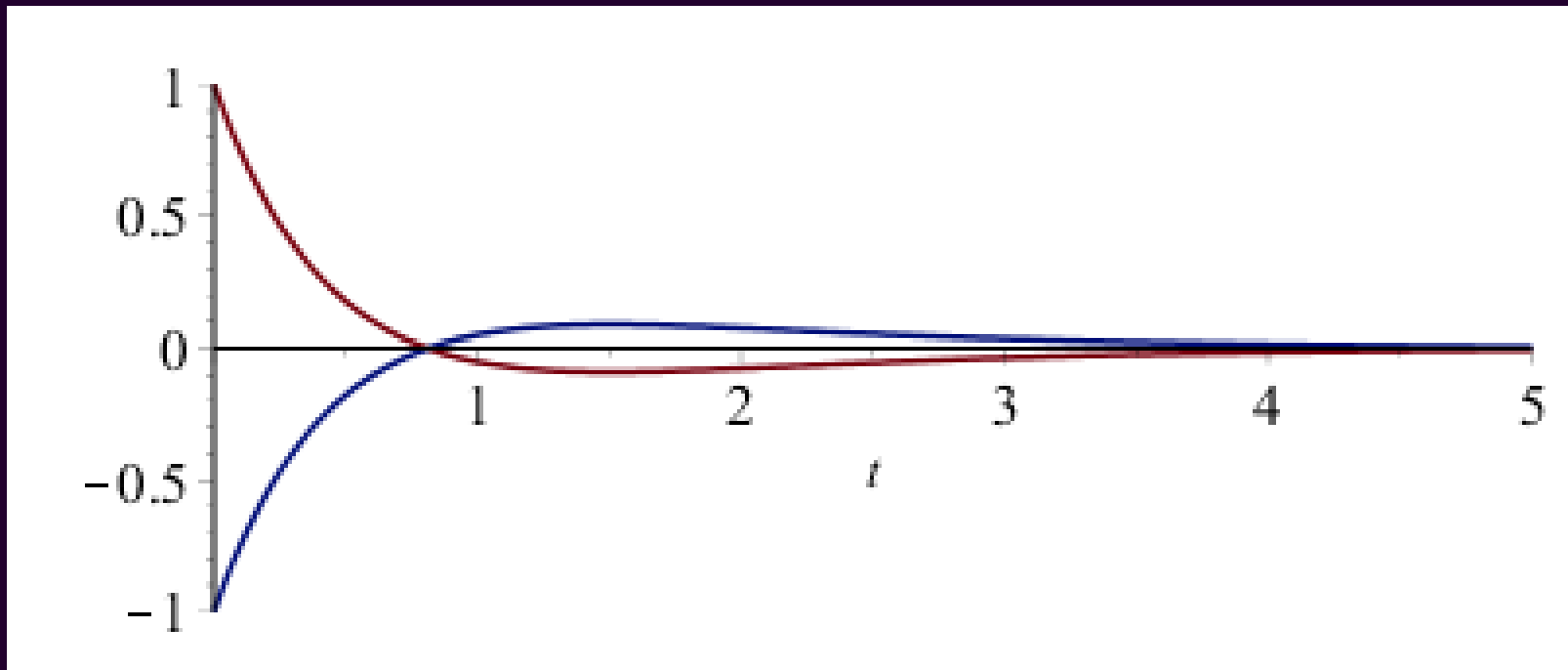
- Here is the sum of two signals $1.8e^{-2t}$ and $-0.8e^{-t}$





Continuous signals

- Here is that sum and its additive inverse





Continuous signals

- Claim:
 - With these definitions of scalar multiplication, vector addition, the zero vector and additive inverses, the set of all continuous signals forms a real or complex vector space





Real or complex?

- Recall that discrete signals may be real or complex:

$$(1, 1, 1, 1, 1, 1, \dots)$$

$$(1, j, -1, -j, 1, j, -1, \dots)$$

- Similarly, polynomial and continuous signals will be functions of a real variable x or t , but their values may be real or complex

$$x^2 + 3x + 1$$

$$x^2 + 5jx + 2 - j$$

$$3.8 e^{-2t}$$

$$7.2 e^{jt}$$

- In general, engineers do not deal with complex domains, only complex ranges





Vectors versus the entries

- Now, many of you may be used to referring a polynomial $p(x)$ or a function $\sin(x)$
 - Please remember, there is a difference between the polynomial or function and its evaluation

Finite-dimensional vectors

$\mathbf{u}$

u_k

Discrete signals

x

$x[k]$

Polynomials

p

$p(x)$

Continuous signals

f

$f(x)$

$\sin$

$\sin(x)$





Images

- A black-and-white image has $m \times n$ pixels,
each of which has an intensity between $[0, 1]$
 - If we remove the restriction on the intensity,
we can think of such an image as a vector
 - For example:
 - We can multiply the intensity of each pixel by a constant α
 - We can add the intensities of two images
 - There is the zero image (all black, or zero)
 - There is a negative image,
where each pixel has the negative value of its intensity
- You may want to think of an image as a matrix,
but it's much more clearly a vector





Images

- Important:
 - Just like the vector spaces of 5-dimensional real vectors and 8-dimensional real vectors are different
 - So too are the vector spaces of 1024×768 pixel images and 3840×2160 pixel images
- If every pixel can take on all real values, you could interpret
 - Any intensity greater than 1 to be white
 - Any negative intensity to be black





Summary

- Following this topic, you now
 - Understand there are many different vector spaces
 - Are aware engineers will use:
 - Discrete signals
 - Polynomials
 - Continuous signals
 - Images





References

- [1] https://en.wikipedia.org/wiki/Vector_space





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

<https://www.rbg.ca/>

for more information.





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